INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, Statistics - III, Compensatory Examination
Time: 3 Hours **Answer all questions** **Marks: 50**

1. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n\times p}$ has rank $r \leq p$ and $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$. If $\hat{\beta}$ is any least squares estimator of β, show that $(\hat{\beta}-\beta)'\mathbf{X}'\mathbf{X}(\hat{\beta}-\beta)$ is distributed independently of the residual sum of squares. [10]

2. Consider the following model:

$$\begin{aligned} y_1 &= \theta + \gamma + \epsilon_1 \\ y_2 &= \theta + \phi + \epsilon_2 \\ y_3 &= 2\theta + \phi + \gamma + \epsilon_3 \\ y_4 &= \phi - \gamma + \epsilon_4, \end{aligned}$$

where ϵ_i are uncorrelated having mean 0 and variance σ^2.
(a) Show that $\gamma - \phi$ is estimable. What is its BLUE?
(b) Find the residual sum of squares. What is its degrees of freedom? [12]

3. Suppose $Y_i \sim N(0, \sigma^2)$, $1 \leq i \leq n$ are independent normally distributed random variables. Let $H_{n\times n}$ be an orthogonal matrix and $\Delta_{n\times n}$ be a diagonal matrix with the first r diagonal entries equal to 1, rest zero, $r \leq n$. Let $C = H'\Delta H$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)'$.
(a) Find $E(\mathbf{Y}'C\mathbf{Y})$.
(b) Find the probability distribution of $\mathbf{Y}'C\mathbf{Y}$. [12]

4. Consider the model:

$$y_{ij} = \mu + \alpha_i + \tau_j + \epsilon_{ij},$$

$1 \leq i \leq 4$, $j = 1, 2$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$ and $0 = \tau_1 + \tau_2 = \sum_{i=1}^4 \alpha_i$.
(a) Show that $\tau_1 - \tau_2$ and $\alpha_k - \alpha_l$, $1 \leq k < l \leq 4$ are estimable.
(b) Find the best linear unbiased estimators of the above mentioned linear contrasts.
(c) Find the variance of the estimators in (b) above and then provide an unbiased estimator for each of these variances. [16]